RELAXATION OF A LIQUID LAYER UNDER THE ACTION OF CAPILLARY FORCES

O. V. Voinov

The theory of creeping motion is used to study the relaxation of an infinite viscous fluid layer (membrane) of nonuniform thickness. The propagation of boundary perturbations in a semi-infinite layer under the action of surface-tension forces is also considered. The layer has at least one common boundary with a gas. It is found that relaxation processes of an infinite layer or the propagation of boundary perturbations inside a bounded layer are nonmonotonic, and that wave-like surface perturbations always arise. Relaxation times are determined. Maximum distances are found over which separate regions of the layer can affect each other.

1. Fundamental Equations. It is assumed that the thickness h of the viscous fluid layer varies over distances l such that $l \gg h$, i.e., $dh/dx \ll 1$ (x is the coordinate in the direction of the layer). We know [1] that the equations of hydrodynamic lubrication theory are valid when the reduced Reynolds' number $R^* \ll 1$ ($R^* = vh^2/l\nu$; v is the velocity along the layer). For small wave-like perturbations, when the variation of thickness $\Delta h \ll h$, this condition is insufficient, since the nonsteady-state term in the Navier-Stokes equation can be large. We must therefore take the more general condition $h^2 \ll \nu\tau$, where τ is the characteristic time for variation of the layer thickness.

The equations of motion and conservation of mass have the form [2]

$$\frac{\partial p}{\partial x} = \rho g + \mu \frac{\partial^2 v}{\partial y^2}, \qquad \frac{\partial}{\partial x}\int_0^h v\,dy + \frac{\partial h}{\partial t} = 0. \tag{1.1}$$

Here y is the coordinate across the layer; y = 0, y = h are the coordinates of the surfaces bounding the layer; g is the mass force.

In the case of a membrane situated on a solid surface, we can assume that a constant shear stress F, applied externally, acts on the free surface of the membrane. Consequently $\mu\,\partial v/\partial y = F$ for $y = h$. Clearly v = 0 and y = 0 also. These boundary conditions and the first of Eqs. (1.1) are satisfied by

$$2v\mu = (\partial p / \partial x - \rho g)(y^2 - 2yh) + 2Fy .$$

If the second equation of (1.1) is taken into account we have

$$\frac{\partial}{\partial x}\left[\frac{h^3}{3\mu}\left(\frac{\partial p}{\partial x} - \frac{3}{2h}F - \rho g\right)\right] = \frac{\partial h}{\partial t} . \tag{1.2}$$

It is known [2] that the boundary condition at the free surface of a fluid can coincide with the boundary condition of a solid body if substances with surface activity are present. In what follows, membranes with this type of boundary condition are referred to as stabilized membranes. It is not difficult to obtain an equation similar to (1.2) for a stabilized layer, if we allow for the fact that the layer suffers only sym-

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metric deformations relative to the center, because the pressure is constant over the cross section and there are surface-tension forces acting. This equation has the form

$$\frac{\partial}{\partial x}\left[\frac{h^3}{12\mu}\left(\frac{\partial p}{\partial x}-\rho g\right)\right]=\frac{\partial h}{\partial t}. \tag{1.3}$$

This equation is treated in [3] for the case in which $\partial p/\partial x = 0$. If it is assumed that the gas pressure at the free surface is constant, the following expressions may be written down for the pressure inside the membrane:

$$(p-p_0)_1=\sigma\partial^2 h/\partial x^2, \qquad (p-p_0)_2={}^1/_2\,\sigma\partial^2 h/\partial x^2. \tag{1.4}$$

Here σ is the surface-tension coefficient; the subscript 1 refers to the membrane on the solid surface; the subscript 2 refers to a stabilized membrane having a boundary with a gas only.

If one of the equations (1.4) is inserted in (1.2) or (1.3) and the result linearized, the following equation is obtained:

$$\frac{\partial h}{\partial t}=-a\frac{\partial^4 h}{\partial x^4}-b\frac{\partial h}{\partial x}. \tag{1.5}$$

For a nonstabilized membrane on a solid surface

$$b\mu={}^3/_2 hF+\rho g h^2, \qquad 3\mu a=\sigma h^3.$$

For a stabilized membrane having a boundary with a gas only

$$4b\mu=\rho g h^2, \qquad 24\mu a=\sigma h^3.$$

If one of the boundaries is a solid body then the coefficient b remains the same, while the coefficient a is doubled.

2. An Infinite Membrane. The Cauchy problem can be correctly formulated for equation (1.5) if $a > 0$, as can be seen from what follows. For $a < 0$ the formulation is incorrect. Since $a > 0$ always for a membrane, the problem can be formulated with the initial condition

$$h=h_0(x) \quad \text{for } t=0,\ -\infty<x<+\infty. \tag{2.1}$$

A Laplace transform with respect to time and a Fourier transform with respect to the space coordinate can be used in order to solve Eq. (1.5) with the initial condition (2.1):

$$h(k,p)=\int_0^\infty dt\int_{-\infty}^{\infty} h(x,t)\,e^{-pt+ikx}dx. \tag{2.2}$$

Equation (1.5) then gives

$$ph(k,p)-h_0(k)=-(ak^4+bik)h(k,p)$$

where $h_0(k)$ is the Fourier transform of the function $h_0(x)$. The Fourier transform of the function $h(x,t)$ can then be found easily:

$$h(k,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}h(k,p)\,dp=h_0(k)\,e^{-(ak^4+bik)t}. \tag{2.3}$$

When the inverse Fourier transform is taken and the convolution theorem used, h can be expressed in terms of h_0 with the help of the Green's function:

$$h(x,t)=\int_{-\infty}^{\infty} h_0(\xi-bt)\,G(x-\xi,t)\,d\xi \tag{2.4}$$

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty}{}' e^{ikx - ak^4t} dk . \tag{2.5}$$

It is clear from (2.4) that the relaxation process in a moving membrane ($b \neq 0$) proceeds in the same way as in a stationary membrane ($b = 0$).

In order to investigate the properties of the Green's function (2.5) it is convenient to introduce the parameter

$$s = 3^{3/2} 2^{-11/3} (x^4 / at)^{1/3} \tag{2.6}$$

The parameter $s \sim 1$, if $x \sim 2(at)^{1/4}$. If $s \ll 1$ (for small distances or large times) then we see from (2.5) that the Green's function decreases with time like $(at)^{-1/4}$.

If $s \gg 1$, then the integral (2.5) should be evaluated by the method of steepest descent. It can be shown that the saddle points $k = 2^{-2/3} e^{1/6\,\pi i}$, $k = 2^{-2/3} e^{5/6\,\pi i}$ are the highest on the integration path represented by the straight line $\operatorname{Im} k = 2^{-5/6}\sqrt{3}$. After straightforward calculations we have for $s \to \infty$

$$G(x, t) = 2^{1/4} \sqrt{\pi}\, (ats\sqrt{3})^{-1/4} \exp(-s / \sqrt{3}) \left[\cos(s - {}^1/_6\, \pi) + O(s^{-1})\right]. \tag{2.7}$$

Consequently the Green's function (2.5) for the problems (1.5, 2.1) is an alternating function. This is its fundamental qualitative difference from the Green's function for the heat conduction equation, which is monotonic. We can conclude that the propagation of an initial perturbation through the membrane is always accompanied by the production of waves. Relaxation due to capillary forces has a nonmonotonic character, i.e., every initial perturbation of thickness, even the smoothest, subsequently gives rise to thickness oscillations of the membrane.

It is clear from (2.4)-(2.7) that the characteristic relaxation time for a membrane with irregularities of dimension l is $\tau = l^4/a$. For layers which are thin enough this time can attain days and even months. For example, it is of the order of days when $l \sim 0.1$ cm, $h \sim 10^{-4}$ cm, $\sigma \sim 10^2$ dyn/cm, $\mu \sim 10^{-2}$ g/cm · sec.

It follows from Eqs. (2.4)-(2.7) that parts of the membrane situated at distances farther apart than $r = (at)^{1/4}$ do not influence each other. If the dimensions of the membrane are much greater than r, then there is no point in studying the membrane as a whole, and we can restrict ourselves to treating the individual parts. For a soap bubble of radius 1 cm, for example, the life time $t \approx 10^2$ sec, the wall thickness $h \sim 10^{-4}$ cm, for $\mu \sim 10^{-2}$ g/cm · sec, $\sigma \sim 10^2$ dyn/cm, $r \sim 2 \cdot 10^{-2}$ cm $\ll$ 1 cm. Consequently when investigating processes taking place in the membrane the entire soap bubble need not be treated, and its surface curvature can even be neglected.

Clearly if the dimensions of the membrane are much greater than r, then the region close to the boundary of the membrane can be considered separately, and the membrane treated as semi-infinite.

3. A Semi-infinite Membrane. The solutions of Eq. (1.5) are now considered for $b = 0$ in the interval $(0, \infty)$. In addition to the initial condition

$$h = h_0(x) \quad \text{for} \quad t = 0,\ 0 < x < \infty \tag{3.1}$$

there are two types of boundary conditions. The first set of boundary conditions corresponds to specifying the mass flux from the membrane and the angle of inclination to the membrane boundary. The second set of boundary conditions corresponds to specifying the pressure and thickness. Boundary conditions of the second type are possible but they are not discussed here. Boundary conditions of the first type are

$$\partial^3 h / \partial x^3 = \alpha(t), \quad \partial h / \partial x = \beta(t) \quad \text{for} \quad x = 0 \tag{3.2}$$

In order to construct a solution of the problem for these boundary conditions we can continue the initial condition (3.1) symmetrically about the origin for $x < 0$ and apply the Laplace and Fourier transforms (2.2). If we then take into account that derivatives of $h(x,t)$ with respect to x are discontinuous at the origin, as specified by (3.2), then the following formula can be obtained for transforming the function $h(x,t)$:

$$(p + ak^4) h(k, p) = h_0(k) + 2a\alpha(p) - 2ak^2\beta(p)$$

where $h_0(k)$ is the Fourier transform of the function equal to $h_0(x)$ for $x > 0$ and $h_0(-x)$ for $x < 0$. Application of the inverse Laplace transform to $h(k,p)$ gives

$$h(k, t) = 2a\int_0^t [\alpha(\tau) - k^2\beta(\tau)]\, e^{-ak^4(t-\tau)} d\tau + h_0(k)\, e^{-ak^4 t},$$

On applying the inverse Fourier transform we have the solution of the problem with the boundary conditions (3.2) in the following form:

$$h(x, t) = \int_0^\infty h_0(\xi)\,[G(x-\xi, t) + G(x+\xi, t)]\, d\xi + 2a\int_0^t \left[G(x, t-\tau)\,\alpha(\tau) + \frac{\partial^2 G}{\partial x^2}(x, t-\tau)\,\beta(\tau)\right] d\tau. \qquad (3.3)$$

Here G(x,t) is determined by equation (2.5).

Specifying the thickness and pressure at the boundary is equivalent to the following conditions:

$$h = \varkappa(t), \quad \partial^2 h / \partial x^2 = \gamma(t) \quad \text{for } x = 0. \qquad (3.4)$$

To solve Eq. (1.5) with the boundary conditions (3.1), $h_0(x)$ must be continued asymmetrically about the origin for $x < 0$ in this case. Carrying out calculations similar to those used to obtain equation (3.3), we can find a solution of the problem with the boundary conditions (3.4) in the following form:

$$h(x, t) = \int_0^\infty h_0(\xi)\,[G(x-\xi, t) - G(x+\xi, t)]\, d\xi + 2a\int_0^t \left[\frac{\partial G}{\partial x}(x, t-\tau)\,\gamma(\tau) + \frac{\partial^3 G}{\partial x^3}(x, t-\tau)\,\varkappa(\tau)\right] d\tau. \qquad (3.5)$$

Here, as in (3.3), G(x,t) is defined by equation (2.5).

It is interesting to determine how perturbations propagate away from the boundary of the membrane for small times or large distances, i.e., for large values of the parameter s, defined by (2.6). When the method of steepest descents is applied to $\partial G/\partial x$ we have for $s \to \infty$

$$\partial G / \partial x = \sqrt{2\pi}\,(3at)^{-1/2} \exp(-s/\sqrt{3})\,[\sin s + O(s^{-1})], \qquad (3.6)$$

Similar equations can easily be written down for $\partial^2 G/\partial x^2$, $\partial^3 G/\partial x^3$. It is important that they contain oscillations, like (3.6). It then follows from (3.3) and (3.5) that perturbations at the boundary cause oscillating perturbations in the membrane at large distances from the boundary.

If α = const, β = const in the boundary conditions (3.2), then it follows from (3.3) that a decrease in thickness occurs most rapidly at the boundary, in accordance with the following law:

$$h - h_0 = -\tfrac{1}{3}\alpha c_1 (at)^{3/4} - \beta c_2 (at)^{1/4}$$

$$\frac{\pi c_1}{4} = \int_{-\infty}^{\infty} e^{-z^4} dz \approx 1.81, \qquad \frac{\pi c_2}{4} = \int_{-\infty}^{\infty} z^2 e^{-z^4} dz \approx 0.60. \qquad (3.7)$$

If a constant pressure perturbation Δp acts at the boundary with a constant thickness (γ = const, $\varkappa$ = const), then a maximum change in thickness Δh occurs at some point $x_0 > 0$. On the basis of (3.5)

$$\Delta h(x_0, t) \approx -\gamma\,(at)^{1/2}, \quad x_0(t) \approx (at)^{1/4}. \qquad (3.8)$$

A membrane of dimension l will become thinner in the boundary region, if $x_0 \ll l$ for $\Delta h \sim h$, i.e., $\Delta p l^2 \gg h\sigma$, which follows from (3.8). If the reverse inequality is satisfied, the membrane will become thinner in the central region.

The assumptions $h^2 \ll \nu\tau$ and $h \gg \Delta h$, made in deriving (1.5), will remain valid for $a\gamma^2 \ll \nu$ and $\gamma\sqrt{at} \ll h$ on the basis of (3.8). For a stabilized membrane having a boundary with a gas only, this means that

$$h^3 (\Delta p)^2 \ll 24\rho\nu^2\sigma, \quad \Delta p\sqrt{ht} \ll 5\sqrt{\mu\sigma}.$$

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